J. DIFFERENTIAL GEOMETRY
12 (1977) 209-227

GAUGE ALGEBRAS, CURVATURE AND SYMPLECTIC STRUCTURE

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Introduction

The notion of "gauge algebra" has its origin in the theory of the electromagnetic field. In the most simple case (vacuum space) a electromagnetic field is defined by a 1-form ω on the Minkowski space V_4 which satisfies the Maxwell equations:

$$\delta d\omega = 0 \ ,$$

where d is the exterior differential, and δ is the codifferential with respect the Minkowski metric g. ω is called the field potential 1-form.

As is known, these equations can be obtained as Lagrange equations of the variational problem defined by the Lagrangian density $\mathscr{L}dx$, where dx is the Minkowski volume element, and $\mathscr{L}$ is the real valued function defined on the 1-jets fibre bundle $J^1(T^*(V_4))$ by

$$\mathscr{L}(j_x^1\omega) = \tfrac{1}{4}g_x(d\omega, d\omega) \ .$$

In this way we have associated a dynamical theory to the electromagnetic field (Hamilton equations, Poisson algebra, etc.). In particular, an important notion to consider is the *Lie algebra of the infinitesimal internal symmetries* of the field, that is, the vertical vector fields D on $T^*(V_4)$ such that their 1-jet extension $j^1(D)$ satisfies the condition $j^1(D)\mathscr{L} = 0$, [1]. In our case, this Lie algebra is the abelian real Lie algebra defined by the infinitesimal generators D_f of the uniparametric groups τ_t of the automorphisms of $T^*(V_4)$ given by

$$\tau_t : \omega_x \mapsto \omega_x + t(df)_x \ ,$$

where f runs along the algebra $\{f\}$ of the real valued differentiable functions on V_4. In this way, at the base of the dynamical theory of electromagnetic field we find a special real Lie algebra $\{f\}$ and a natural representation $f \in \{f\} \to D_f$ of this algebra in the vector fields on the space $T^*(V_4)$. This is the *gauge algebra* in the electromagnetic field theory.

The above formulation gives a very interesting geometric insight which as is

Communicated by B. Kostant, May 19, 1975.

proved in [6] corresponds in physics to the fact that *an electromagnetic field is the radiation field generated by a moving electric particle*. Precisely, an electric particle is characterized by a variational problem defined on a fibre bundle $B = V_4 \times F$ (which is the direct product of the Minkowski space V_4 with a real vector space F) which admits the unitary group $U(1)$ as a subgroup of the *group of internal symmetries*. The corresponding Noether invariant is called the *charge-current* 3-*form* of the electric particle. Note that B is associated to the principle bundle $P = V_4 \times U(1)$, whose connections are identified precisely with the 1-forms on V_4 on which the electromagnetic theory has been built. In this way, one has the following natural equivalences: "electromagnetic fields" $\leftrightarrow$ "connections"; "Lagrangian of the field" $\leftrightarrow$ "function of the curvature"; "Gauge algebra" $\leftrightarrow$ "sections of the adjoint fiber bundle of P", etc.

All this leads us to define the notion of gauge algebra of an arbitrary principal bundle $p : P \rightarrow V$ as *the Lie algebra of sections of its adjoint fibre bundle* $L(P)$. The object of this paper is now the following.

After defining a canonical action of the so defined gauge algebra on the connections of the principal bundle, which locally agrees with the formulas suggested by the physiciens [9], we study the relation between the notions of gauge algebra and curvature. The following are two main results in this sense.

First, the principal bundle $\bar{p} : \bar{P} \rightarrow E$ induced from $p : P \rightarrow V$ on its fibre bundle of connections $\pi : E \rightarrow V$ by the projection π has a canonical connection whose curvature 2-form Ω defines a special symplectic structure on E such that the gauge algebra is identified with a certain subalgebra of the corresponding Poisson algebra. According to this: *every gauge algebra is a subalgebra of a Poisson algebra in a cannonical way*. One gets to this result adapting adequately the idea of "pre-quantization" introduced by B. Kostant for ordinary symplectic manifolds [4]. This result is not only interesting in itself, as it relates to apparently different notions like *gauge algebras* and *Poisson algebras*, but opens- the author thinks- the posibility of applying the ideas on "pre-quantization" and "quantization" to the study of unitary representations of gauge algebras.

A second main result is an intrinsic characterization of a known result of Utiyama about "admisible lagrangians" in the gauge-invariant classical field theories [8].

Finally, we apply the obtained results to the problem of "combination" of gauge algebras with the so-called "infinitesimal external symmetries" in classical field theory. Remarks made in this sense can be a good starting point for a differential-geometric approach to this interesting topic for infinite-dimensional Lie algebras of the type of those dealt with in this paper.

Concepts and notation in this paper are the ones usually found in any text on modern differential geometry. The reader can refer to the book by J. L. Koszul [5]. All manifolds will be considered paracompact and connected. Differentiability will always mean C^∞-differentiability, etc.

The author wishes to acknowledge his indebtedness to Professor J. Sancho

for his valuable orientations and effective help and, above all, for his constant and sharp criticism during the preparation of this paper.

1. The fibre bundle of connections of a principal bundle

Let $p: P \to V$ be a principal bundle with structural group G with Lie algebra $\mathscr{G}$. As it is known [5] that a connection on P can be defined by a *splitting* $\sigma: T \to Q$ of the exact sequence of vector bundles on V:

$$(1.1) \qquad 0 \longrightarrow L(P) \xrightarrow{i} Q \xrightarrow{p^*} T \longrightarrow 0$$

where Q is the vector bundle of G-invariant vector fields on P, $L(P)$ is the subbundle of Q defined by the G-invariant vector fields which are tangent to the fibers of P, and T is the tangent bundle of V.

$L(P)$ is a bundle of Lie algebras, where, if $D, D' \in L(P)_x$, then $[D, D']$ is the Lie bracket of D and D'. On the other hand, it is the fibre bundle associated with P by the adjoint representation of G. It is called the *adjoint bundle* of P.

Thus connections of P can be identified with global sections of the affine bundle $\pi: E \to V$ defined as follows: $x \in V$ being given, let E_x be the set of homomorphisms $\sigma_x: T_x \to Q_x$ such that $p_x^* \cdot \sigma_x = 1$, let $E = \bigcup_{x \in V} E_x$ and let π be the natural projection of E onto V.

Proposition 1.1. *$\pi: E \to V$ has a unique affine bundle structure such that for every connection σ on P the mapping σ:* Hom $(T, L(P)) \to E$ *defined by $h_x \mapsto \sigma(x) + h_x$ is an affine bundle isomorphism on V.*

Proof. A connection σ on P being given, the above said mapping is bijective and makes the following diagram commutative:

$$\begin{array}{ccc} \operatorname{Hom}(T, L(P)) & \xrightarrow{\sigma} & E \\ & \searrow \quad \swarrow {\scriptstyle \pi} & \\ & V & \end{array}$$

Then the affine bundle structure of Hom $(T, L(P))$ defines, by σ, an affine bundle structure on E which, we will see, does not depend on the connection σ chosen. Indeed, let σ' be another connection. Then $\sigma'^{-1} \cdot \sigma$: Hom $(T, L(P)) \to$ Hom $(T, L(P))$ is the affine bundle automorphism:

$$h_x \mapsto (\sigma - \sigma')(x) + h_x$$

which proves the desired result. q.e.d.

Let $F(E)$ be the vertical bundle of E, i.e., the subbundle of the tangent bundle of E defined by the vectors tangent to the fibres of E.

Corollary 1. *There is a canonical vector bundle isomorphism on E between the vertical bundle $F(E)$ of E and the vector bundle π^** Hom $(T, L(P))$ *induced of* Hom $(T, L(P))$ *by π.*

Proof. $h_x \in \text{Hom}(T, L(P))_x$ being given, let D_{h_x} be the infinitesimal generator of the uniparametric group τ_t of automorphisms of the fibre E_x:

$$\tau_t(\sigma_x) = \sigma_x + th_x , \qquad \sigma_x \in E_x .$$

The mapping which assigns to each $(\sigma_x, h_x) \in \pi^* \text{Hom}(T, L(P))$ the element $(D_{h_x})_{\sigma_x} \in F(E)$ is the desired isomorphism.

Corollary 2. *E is an affine subbundle of the vector bundle* Hom (T, Q).

Proof. A connection σ on P being given, it is enough to remark that the isomorphism of Prop. 1.1 is the restriction to the subbundle $\text{Hom}(T, L(P)) \subset \text{Hom}(T, Q)$ of the affine bundle automorphism $a_x \mapsto \sigma(x) + a_x$ of Hom (T, Q).

Definition 1.1. The affine bundle E will be called the *fibre bundle of connections* of the given principal bundle P.

2. Gauge algebra of a principal bundle and its natural representation on the fibre bundle of connections

Let A be the real algebra of the real valued differentiable functions on V.

Definition 2.1. The Lie A-algebra Γ of global sections of the adjoint bundle $L(P)$ will be called the *gauge algebra* of the principal bundle P.

Examples. (1) If G is abelian, then $\mathcal{G}$ is also abelian and $L(P)$ can be identified with the trivial bundle $V \times \mathcal{G}$. Thus the gauge algebra is just the abelian Lie algebra of $\mathcal{G}$-valued differentiable functions on V. In particular, if $G = U(1)$, then $\mathcal{G} = R$ and $\Gamma = A$, which is the gauge algebra in the electromagnetic field theory.

(2) If $P = V \times G$, then $L(P) = V \times \mathcal{G}$, so Γ can be identified with the tensor product $A \otimes \mathcal{G}$ endowed with the Lie product:

$$[f \otimes e, f' \otimes e'] = (f \cdot f') \otimes [e, e'] ,$$

where $f, f' \in A$ and $e, e' \in \mathcal{G}$. One has the so-called "current algebras" introduced by M. Gell-Mann [3].

(3) The sheaf of sections of $L(P)$ gives us a family of gauge algebras (parametrized by the open sets of V): for every open set $U \subset V$, Γ_U is the gauge algebra of the principal bundle P_U.

Every element s of the gauge algebra Γ defines an uniparametric group τ_t of the vertical automorphisms of the fibre bundle of connections E in the natural way:

$$\tau_t \sigma_x = \sigma_x + t[\sigma_x, s] , \qquad \sigma_x \in E ,$$

where $[\sigma_x, s] \in \text{Hom}(T, L(P))$ is defined by

$$[\sigma_x, s] D_x = [\sigma_x(D_x), s] .$$

By the canonical isomorphism between $F(E)$ and π^* Hom $(T, L(P))$ (Cor. 1, Prop. 1.1), the infinitesimal generator D_s of τ_t is the vertical vector field on E:

$$D_s : \sigma_x \mapsto [\sigma_x, s] \ .$$

Theorem 2.1. *The mapping $s \in \Gamma \mapsto D_s$ is a homomorphism of real Lie algebras.*

Proof. τ_t is the restriction to $E \subset$ Hom (T, Q) of the uniparametric group $\bar{\tau}_t$ of the vertical automorphisms of Hom (T, Q):

$$\bar{\tau}_t a_x = a_x + t[a_x, s] \ , \qquad a_x \in \text{Hom}\,(T, Q) \ ,$$

where $[a_x, s] \in$ Hom $(T, L(P))_x$ is defined by

$$[a_x, s] D_x = [a_x(D_x), s] \ .$$

Thus D_s is the restriction to E of the infinitesimal generator $\bar{D}_s$ of $\bar{\tau}_t$.

Accordingly, the theorem would follow automatically if $s \in \Gamma \mapsto \bar{D}_s$ were a homomorphism of real Lie algebras. We shall see that it is the case.

Linearity is immediate. To prove the equality $\bar{D}_{[s,s']} = [\bar{D}_s, \bar{D}_{s'}]$ it will be enough to prove it on functions f of Hom (T, Q) linear on the fibres, because the $\bar{D}_s$ are vertical. Since for these functions $(\bar{D}_s f)(a_x) = f([a_x, s])$ (it follows that, in particular, the $\bar{D}_s f$ are also linear on the fibres), the following calculation proves what we want:

$$\begin{aligned}(\bar{D}_{[s,s']} f)(a_x) &= f([a_x, [s, s']]) = f([[a_x, s], s']) - f([[a_x, s'], s]) \\ &= (\bar{D}_{s'} f)([a_x, s]) - (\bar{D}_s f)([a_x, s']) \\ &= (\bar{D}_s(\bar{D}_{s'} f))(a_x) - (\bar{D}_{s'}(\bar{D}_s f))(a_x) \\ &= ([\bar{D}_s, \bar{D}_{s'}] f)(a_x) \ . \qquad \text{q.e.d.}\end{aligned}$$

Theorem 2.1 gives us a representation of gauge algebras (by vector fields on a manifold) which we shall call, in what follows, *the natural representation of the gauge algebra Γ of P on the fibre bundle of connections $\pi : E \to V$.*

Local expression. If U is an open set of V with local coordinates (x_i) such that $P_U \approx U \times G$ and (D_j) are the G-invariant vector fields on P_U defined by a basis of the Lie algebra $\mathcal{G}$ in the corresponding isomorphism $L(P)_U \approx U \times \mathcal{G}$, then the functions $(x_i A_{ij})$ on E_U, where

$$\sigma_x\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i} + \sum_j A_{ij}(\sigma_x) D_j \ , \qquad \sigma_x \in E_U \ ,$$

define a system of local coordinates on $E_U \subset E$.

On the other hand, the gauge algebra Γ_U can be identified with the A_U-module of linear combinations

$$s = \sum_j f_j(x_i)D_j \; ,$$

endowed with the Lie product

$$[s, s'] = \sum_{i,j} f_i \cdot f_j [D_i, D_j] = \sum_{i,j,k} f_i \cdot f_j c_{ij}^k D_k \; ,$$

where (c_{ij}^k) are the structural constants of $\mathscr{G}$. In this setting, the vector field $D_s : \sigma_x \mapsto [\sigma_x, s]$ associated to s can be calculated as follows:

$$[\sigma_x, s]\frac{\partial}{\partial x_i} = \left[\sigma_x\left(\frac{\partial}{\partial x_i}\right), s\right] = \left[\frac{\partial}{\partial x_i} + \sum_h A_{ih}(\sigma_x)D_h, \sum_k f_k D_k\right]$$
$$= \sum_j \left(\left(\frac{\partial f_j}{\partial x_i}\right)_x + \sum_{h,k} c_{hk}^j A_{ih}(\sigma_x) f_k(x)\right) D_j \; ,$$

from which it follows that

$$D_s = \sum_{i,j} \left(\frac{\partial f_j}{\partial x_i} + \sum_{h,k} c_{hk}^j A_{ih} f_k\right) \frac{\partial}{\partial A_{ij}} \; . \tag{2.1}$$

3. A symplectic characterization of gauge algebras by means of curvature

Let $\bar{p} : \bar{P} \to E$ be the induced bundle of the principal bundle $p : P \to V$ on its fibre bundle of connections $\pi : E \to V$ by the projection π. It is a principal bundle with structural group G such that the canonical morphism $\bar{\pi} : \bar{P} \to P$ is a principal G-bundle morphism, i.e., one has the following commutative diagram:

$$\begin{array}{ccc} \bar{P} & \xrightarrow{\bar{\pi}} & P \\ \downarrow & & \downarrow \\ E & \xrightarrow{\pi} & V \end{array}$$

where $\bar{\pi}$ commutes with the action of G.

In this way, if one considers the exact sequence (1.1) corresponding to $\bar{p} : \bar{P} \to E$, then $\bar{\pi}$ induces a morphism $f : \bar{Q} \to Q$ of vector bundles, which in turn induces a morphism of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L(\bar{P}) & \longrightarrow & \bar{Q} & \longrightarrow & \bar{T} & \longrightarrow & 0 \\ & & f\downarrow & & f\downarrow & & \pi^*\downarrow & & \\ 0 & \longrightarrow & L(P) & \longrightarrow & Q & \longrightarrow & T & \longrightarrow & 0 \end{array}$$

We want to remark that $L(\bar{P})$ can be identified with the vector bundle $\pi^*L(P)$ induced of $L(P)$ by π, after $f: L(\bar{P}) \to L(P)$ coincides with the corresponding canonical morphism. Thus $f|_{L(\bar{P})}$ is an isomorphism on each fibre.

Then the exact sequence

$$0 \to L(\bar{P}) \to \bar{Q} \to \bar{T} \to 0$$

has a "canonical splitting" $\rho: \bar{Q} \to L(\bar{P})$ defined by

$$\rho_{\sigma_x}(\bar{D}) = \rho_x(f\bar{D}) \,, \qquad \sigma_x \in E \,,$$

where ρ_x is the projector $1 - \sigma_x \cdot p_x^*$, and $\rho_x(f\bar{D}) \in L(P)_x$ is considered as an element of the fibre $L(\bar{P})_{\sigma_x}$ by the isomorphism $f: L(\bar{P})_{\sigma_x} \to L(P)_x$ which we mentioned before.

Definition 3.1. We shall call *canonical connection* of the principal bundle $\bar{P}$ the connection defined on $\bar{P}$ by the splitting ρ. The corresponding connection 1-form will be written θ.

This connection defines a derivation law ∇ in the Lie module $\Gamma(L(\bar{P}))$ of sections of $L(\bar{P})$. Thus we have an $L(\bar{P})$*-valued differential calculus* on the manifold E. In what follows we shall use this calculus without explicitly mentioning the derivation law ∇.

Local expression. Let $(x_i A_{ij})$ be the system of local coordinates on $E_U \subset E$ defined in § 2. By the identification of $L(\bar{P})_{E_U}$ with the induced vector bundle $\pi^*L(P)_U$, the basis (D_j) of $\Gamma(L(P)_U)$ in § 2 defines a basis of $\Gamma(L(\bar{P})_{E_U})$. A simple local calculation gives for $(\partial^\nabla/\partial x_i)D_j$ and $(\partial^\nabla/\partial A_{ik})D_j$ the expressions:

$$\frac{\partial^\nabla}{\partial x_i} D_j = \sum_{h,k} c^k_{hj} A_{ih} D_k \,, \qquad \frac{\partial^\nabla}{\partial A_{ik}} D_j = 0 \,. \tag{3.1}$$

Now, let Ω be the curvature 2-form of the canonical connection. It is an $L(\bar{P})$-valued 2-form on the manifold E, whose local expression is, by (3.1),

$$\Omega = \sum_{i,j} \left(dA_{ij} \wedge dx_i - \frac{1}{2} \sum_{l,h,k} c^j_{hk} (A_{lh} A_{ik} - A_{ih} A_{lk}) dx_l \wedge dx_i \right) \circ D_j \,. \tag{3.2}$$

Remark. By what was said in § 1, connections on P are identified with global sections of the fibre bundle $\pi: E \to V$. Now one observes that the curvature 2-form Ω has the following universal property: *for every connection* $\sigma: V \to E$ *with curvature 2-form* Ω^σ *one has* $\Omega^\sigma = \sigma^*\Omega$. In particular, one can obtain from here a simple proof of Weil's theorem on characteristic classes [2].

Proposition 3.1. *Ω is an $L(\bar{P})$-valued pre-symplectic metric on the manifold E.*

Proof. From the local expression (3.2) it follows immediately that Ω is nonsingular in every point of E. q.e.d.

Ω is not closed in general. But, if one considers it as an End $L(\bar{P})$-valued 2-form by the rule;

$$\Omega(D, D')s = [\Omega(D, D'), s] \ ,$$

it becomes closed, for then it coincides with the curvature 2-form of the derivation law ∇, which is closed by Bianchi's identity.

In what follows, by abuse of language, we shall consider (E, Ω) as a *symplectic manifold*. We will see that this is justified for the ordinary notions of symplectic manifolds can be generalized to (E, Ω) in a natural way.

By means of the identification of $L(\bar{P})$ with the induced vector bundle $\pi^*L(P)$, the gauge algebra Γ of P is injected onto a A-subalgebra of the Lie algebra $\Gamma(L(\bar{P}))$. Under these conditions we have the following.

Theorem 3.1. *If $s \in \Gamma \mapsto D_s$ is the natural representation of the gauge algebra Γ on the fibre bundle of connections $\pi : E \to V$, then*

$$iD_s\Omega = ds \ ,$$

i.e., D_s is the hamiltonian vector field of (E, Ω) corresponding to s. Γ is characterized as the set of sections $s \in \Gamma(L(\bar{P}))$ with a hamiltonian vector field D_s which is tangent to the fibres of the morphism π.

Proof. By using the local expressions derived in § 2 and § 3, one has

$$\begin{aligned} iD_s\Omega &= \sum_{i,j} iD_s(dA_{ij} \wedge dx_i) \circ D_j = \sum_{i,j} (D_sA_{ij})dx_i \circ D_j \\ &= \sum_{i,j} \left(\frac{\partial f_j}{\partial x_i} + \sum_{h,k} c^j_{hk} A_{ih} f_k \right) dx_i \circ D_j = d \sum_j f_j \circ D_j = ds \ , \end{aligned}$$

where it is supposed that the local expression for s is $s = \sum_j f_j \circ D_j$. If $s \in \Gamma$, we have just seen that it has a hamiltonian vector field D_s tangent to the fibres of π. Conversely, if $s \in \Gamma(L(\bar{P}))$ has a hamiltonian vector field D_s tangent to the fibres of π, then

$$ds = iD_s\Omega = \sum_{i,j} iD_s(dA_{ij} \wedge dx) \circ D_j = \sum_{i,j} (D_sA_{ij})dx_i \circ D_j \ ,$$

so s has the local expression $s = \sum_j f_j(x_i) \circ D_j$, thus proving that $s \in \Gamma$.

Corollary. *The kernel of the representation $s \in \Gamma \mapsto D_s$ is the ideal Γ_0 of sections $s \in \Gamma$ such that $ds = 0$. Γ_0 is locally isomorphic with the center of the Lie algebra $\mathscr{G}$ of the structural group G. In particular, we have two extreme cases: if G is abelian Γ_0 is globally isomorphic with $\mathscr{G}$, and if $\mathscr{G}$ has no center, the representation $s \in \Gamma \mapsto D_s$ is faithful.*

Proof. The first part is an immediate consequence of the theorem. Now, if $s = \sum_j f_j \circ D_j$ on $U \subset V$ and (g_j) is the basis of $\mathscr{G}$ defining the (D_j) (local expression, § 2), then $ds = 0$ is equivalent, by (3.1), to the system of equations

$$\frac{\partial f_j}{\partial x_i} + \sum_{h,k} c^j_{hk} A_{ih} f_k = 0 \ .$$

Taking the derivative with respect to A_{ih} one has $\sum_k c^j_{hk} f_k = 0$, from which it follows that $\partial f_j / \partial x_i = 0$. Then $s \in (\Gamma_0)_U$ if and only if $s = \sum_j \lambda_j D_j$, where the λ_j are real numbers such that $\sum_j c^k_{hj} \lambda_j = 0$.

The mapping $s = \sum_j \lambda_j D_j \in (\Gamma_0)_U \mapsto \sum_j \lambda_j g_j \in \mathscr{G}$ establishes the required (local) isomorphism between Γ_0 and the center of $\mathscr{G}$. Now the last part of the corollary is immediate.

4. Poisson algebra associated to a gauge algebra and prequantization

In § 3 we have seen how the gauge algebra Γ can be injected canonically into the Lie algebra $\Gamma(L(\bar{P}))$ in which the differential calculus on the symplectic manifold (E, Ω) is valued. Moreover, Γ is injected into the A-subalgebra $\bar{\Gamma}$ of $\Gamma(L(\bar{P}))$ defined by the sections $s \in \Gamma(L(\bar{P}))$ which have a hamiltonian vector field. Thus we have the canonical inclusions of Lie A-algebras

$$\Gamma \subset \bar{\Gamma} \subset \Gamma(L(\bar{P})) .$$

$\Gamma \subset \bar{\Gamma}$ is always strict, and $\bar{\Gamma} \subset \Gamma(L(\bar{P}))$ is strict if $\dim G > 1$.

Now on $\bar{\Gamma}$ we should define the notion of "Poisson bracket". We shall see that this can be done in such a way that while preserving all the essential properties of the ordinary Poisson bracket, on Γ the new product coincides with the old one. In particular, it follows that *every gauge algebra can be considered in a canonical way as a subalgebra of a Poisson algebra.*

The method to follow will be a special adaptation of the idea of "prequantization" introduced by B. Kostant for ordinary symplectic manifolds. In this sense, we shall proceed as follows.

The canonical connection of $\bar{p}: \bar{P} \to E$ establishes an isomorphism $\eta: \Gamma(L(\bar{P})) \oplus \mathscr{D} \to \mathscr{L}(\bar{P})$ between the direct sum $\Gamma(L(\bar{P})) \oplus \mathscr{D}$ of the modules $\Gamma(L(\bar{P}))$ of sections of $L(\bar{P})$ and $\mathscr{D}$ of vector fields on E, and the module $\mathscr{L}(\bar{P})$ of G-invariant vector fields on $\bar{P}$, by the rule:

$$\eta(s, D) = -s + \tilde{D} ,$$

where $\tilde{D}$ is the horizontal lift of $D \in \mathscr{D}$, and $s \in \Gamma(L(\bar{P}))$ is a G-invariant vector field on $\bar{P}$ tangent to the fibres of $\bar{p}$.

Denoting by $\bar{p}^*$ the canonical injection of the $L(\bar{P})$-valued forms on E into the $\mathscr{G}$-valued forms on $\bar{P}$ [5] and remembering that we call θ and Ω, respectively, the connection 1-form and the curvature 2-form (as a $L(\bar{P})$-valued 2-form on E) of the canonical connection of $\bar{P}$, we have the following:

Lemma 4.1. *If $s \in \Gamma(L(\bar{P}))$ and $D \in \mathscr{D}$, then*

$$L_{\eta(s,D)}\theta = \bar{p}^*(iD\Omega - ds) .$$

Proof. By putting $\eta = \eta(s, D) = -s + \tilde{D}$, we shall compute the Lie derivative $L_\eta\theta = i\eta d\theta + di\eta\theta$. Denoting, as it is usual [5], $\tilde{s}$ for $\bar{p}^*s$, from $i\eta\theta = \theta(\eta) = \theta(-s) = -\tilde{s}$ we obtain that

$$di\eta\theta = -d\tilde{s} = -\bar{p}^*ds + [\theta, \tilde{s}] .$$

On the other hand, by the structure equation $d\theta = \bar{p}^*\Omega - [\theta, \theta]$ one has

$$i\eta d\theta = i\tilde{D}\bar{p}^*\Omega + is[\theta, \theta] = \bar{p}^*(iD\Omega) - [\theta, \tilde{s}] ,$$

so that

$$L_\eta\theta = i\eta d\theta + di\eta\theta = \bar{p}^*(iD\Omega - ds) . \quad \text{q.e.d.}$$

$\mathscr{L}(\bar{P})$ is a real Lie algebra with respect to the Lie bracket of vector fields. This Lie product is expressed with respect to the above parametrization η as follows.

Lemma 4.2. *If $s_i \in \Gamma(L(\bar{P}))$ and $D_i \in \mathscr{D}$, $i = 1, 2$, then*

$$[\eta(s_1 D_1), \eta(s_2 D_2)] = \eta(D_1 s_2 - D_2 s_1 + \Omega(D_1, D_2) + [s_1, s_2], [D_1, D_2]) .$$

Proof. Let $\eta_i = \eta(s_i D_i)$ and $[\eta_1, \eta_2] = \eta(s, D) = \eta$. Of course, $D = [D_1, D_2]$ since $D = \bar{p}\eta = \bar{p}[\eta_1, \eta_2] = [\bar{p}\eta_1, \bar{p}\eta_2] = [D_1, D_2]$.

On the other hand $\theta([\eta_1, \eta_2]) = -\tilde{s}$. Then by the structure equation $d\theta = \bar{p}^*\Omega - [\theta, \theta]$ one has

$$d\theta(\eta_1, \eta_2) = (\bar{p}^*\Omega)(\tilde{D}_1, \tilde{D}_2) - [\tilde{s}_1, \tilde{s}_2] ,$$

so that

$$\tilde{s} = -\theta([\eta_1, \eta_2]) = \widetilde{\Omega(D_1, D_2)} - \eta_1\theta(\eta_2) + \eta_2\theta(\eta_1) - [\tilde{s}_1, \tilde{s}_2] .$$

Thus from $\tilde{s}_i = -\theta(\eta_i)$ and the definition of covariant derivate it follows that

$$\tilde{s} = -\theta([\eta_1, \eta_2]) = \widetilde{\Omega(D_1, D_2)} + \widetilde{D_1 s_2} + [\tilde{s}_1, \tilde{s}_2] - \widetilde{D_2 s_1} - [\tilde{s}_2, \tilde{s}_1] - [\tilde{s}_1, \tilde{s}_2] .$$

Now we have the required result by considering that the injection $s \mapsto \tilde{s}$ preserves the Lie product.

Corollary. *If $L_{\eta(s_i, D_i)}\theta = 0$, then*

$$[\eta(s_1, D_1), \eta(s_2, D_2)] = \eta([s_1, s_2] - \Omega(D_1, D_2), [D_1, D_2]) .$$

Proof. Obvious after Lemmas 4.1 and 4.2. q.e.d.

Now we can state the most important result in this paragraph:

Theorom 4.1. *Let $\mathscr{L}(\bar{P}, \theta)$ and $\mathscr{H}$ be respectively the real Lie algebras of vector fields $\eta \in \mathscr{L}(\bar{P})$, such that $L_\eta\theta = 0$, and of hamiltonian vector fields of (E, Ω).*

(a) *One has the central extension of real Lie algebras:*

$$0 \to \Gamma_0 \to \mathscr{L}(\bar{P}, \theta) \to \mathscr{H} \to 0$$

where Γ_0 is the kernel of the natural representation $s \in \bar{\Gamma} \mapsto D_s$ of the gauge algebra Γ(Cor. of Th. 3.1), $\Gamma_0 \to \mathscr{L}(\bar{P}, \theta)$ is the injection $s \in \Gamma_0 \mapsto -s$, and $\mathscr{L}(\bar{P}, \theta) \to \mathscr{H}$ is defined by the projection $\bar{p}: \bar{P} \to E$.

(b) *The mapping $\tilde{\delta}: \bar{\Gamma} \to \mathscr{L}(\bar{P}, \theta)$ defined by*

$$\tilde{\delta}(s) = \eta(s, D_s)$$

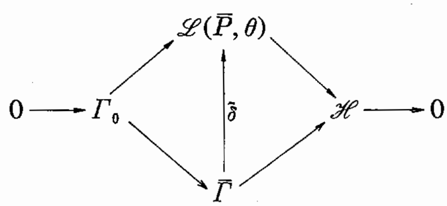
is an isomorphism of real vector spaces.

This allows us to endow $\bar{\Gamma}$ with the Lie product $\{\ ,\ \}$ induced by the isomorphism $\tilde{\delta}$. The real Lie algebra thus defined $(\bar{\Gamma}, \{\ ,\ \})$ will be called *Poisson algebra associated to the gauge algebra Γ*. The Poisson product $\{\ ,\ \}$ is given by

$$\{s, s'\} = [s, s'] - \Omega(D_s, D_{s'}) \ , \tag{4.1}$$

where $D_s, D_{s'}$ are the hamiltonian vector fields corresponding to $s, s' \in \bar{\Gamma}$. In particular, on Γ both products $[\ ,\ ]$ and $\{\ ,\ \}$ coincide.

(c) *One has the commutative diagram of real Lie algebras:*



where $\Gamma_0 \to \bar{\Gamma}$ is the inclusion $\Gamma_0 \subset \bar{\Gamma}$, and $\bar{\Gamma} \to \mathscr{H}$ is the mapping which assigns to every $s \in \bar{\Gamma}$ its corresponding hamiltonian vector field D_s.

Thus the Poisson algebra $\bar{\Gamma}$ is equivalent to the real Lie algebra $\mathscr{L}(\bar{P}, \theta)$ as a central extension of $\mathscr{H}$ by Γ_0. In particular, the gauge algebra Γ is an extension, by Γ_0, of the hamiltonian vector fields tangent to the fibres of $\pi: E \to V$.

Proof. (a) If $\eta = \eta(s, D) \in \mathscr{L}(\bar{P}, \theta)$ then, by Lemma 4.1, $iD\Omega = ds$, from which it follows that $\bar{p}\eta = D \in \mathscr{H}$. The mapping $\mathscr{L}(\bar{P}, \theta) \to \mathscr{H}$ is onto, for, if $D \in \mathscr{H}$, then there exists a section $s \in \Gamma(L(\bar{P}))$ such that $iD\Omega = ds$, from which we have $\eta = \eta(s, D) \in \mathscr{L}(\bar{P}, \theta)$ and $\bar{p}\eta = D$. s is determined up to a section s_0 such that $ds_0 = 0$, i.e., up to an element of Γ_0, thus proving the exactness of the sequence. Mappings are obviously homomorphisms of real Lie algebras. Last, $\Gamma_0 \to \mathscr{L}(\bar{P}, \theta)$ is central by Cor. of Th. 3.1 and Cor. of Lemma 4.2.

(b) It is immediate that $\tilde{\delta}$ is a homomorphism of real vector spaces. That $\tilde{\delta}(s) = 0$ implies $s = 0$ is obvious also. On the other hand, if $\eta = \eta(s, D) \in \mathscr{L}(\bar{P}, \theta)$ then $iD\Omega = ds$, from which we have $s \in \bar{\Gamma}$ and $D_s = D$, i.e., $\tilde{\delta}(s) = \eta$. Thus $\tilde{\delta}$ is an isomorphism.

By the definition of $\{\ ,\ \}$ and Cor. of Lemma 4.2 one has

$$\begin{aligned}\{s, s'\} &= \tilde{\delta}^{-1}[\eta(s, D_s), \eta(s, D_{s'})] \\ &= \tilde{\delta}^{-1}\eta([s, s'] - \Omega(D_s, D_{s'}), [D_s, D_{s'}]) \\ &= [s, s'] - \Omega(D_s, D_{s'}) \ .\end{aligned}$$

On Γ both products $[\ ,\ ]$ and $\{\ ,\ \}$ coincide for, if $s, s' \in \Gamma$, then $\Omega(D_s, D_{s'}) = 0$ because $D_s, D_{s'}$ are tangent to the fibres of $\pi: E \to V$.

Remark. From the preceding theorem one has immediately that if $D_s, D_{s'}$ are the hamiltonian vector fields of $s, s' \in \bar{\Gamma}$, then

$$i[D_s, D_{s'}]\Omega = d\{s, s'\} \ ,$$

that is, $[D_s, D_{s'}]$ is the hamiltonian vector field corresponding to $\{s, s'\}$. In particular, if $s, s' \in \Gamma$ then $i[D_s, D_{s'}]\Omega = d[s, s']$. This gives us a new proof that $s \in \Gamma \to D_s$ is a representation of real Lie algebras.

The Poisson algebra $\bar{\Gamma}$ can be now pre-quantized as in the ordinary case [4]. Let $\delta: \bar{\Gamma} \to \mathrm{End}_R\ (L(\bar{P}))$ defined by

$$\delta(s)r = [s, r] + D_s r \ ,$$

where D_s is the hamiltonian vector field of $s \in \bar{\Gamma}$ and $r \in \Gamma(L(\bar{P}))$.

Theorem 4.2. *δ is a representation of the Poisson algebra $\bar{\Gamma}$ on the real vector space $\Gamma(L(\bar{P}))$, that is, $\delta\{s, s'\} = \delta(s)\cdot\delta(s') - \delta(s')\delta(s)$. Moreover, for every $r \in \Gamma(L(\bar{P}))$ one has*

$$\widetilde{\delta(s)r} = \tilde{\delta}(s)\tilde{r} \ . \tag{4.2}$$

Proof. The following calculus gives (4.2):

$$\tilde{\delta}(s)\tilde{r} = \eta(s, D_s)\tilde{r} = (-s + \tilde{D}_s)\tilde{r} = [\tilde{s}, \tilde{r}] + \widetilde{D_s r} = \widetilde{[s, r] + D_s r} = \widetilde{\delta(s)r} \ .$$

It follows immediately from here that δ is a representation, by observing that $\Gamma(L(\bar{P})) \to \Gamma(\widetilde{L(\bar{P})})$ is an isomorphism and that $\tilde{\delta}$ is a homomorphism of real Lie algebras by Theorem 4.1. q.e.d.

In particular, δ induces a representation of the gauge algebra Γ on the real vector space $\Gamma(L(\bar{P}))$, whose local expression is

$$\delta(s)r = [s, r] + \sum_{i,j,l} \left(\frac{\partial f_j}{\partial x_i} + \sum_{h,k} c^j_{hk} A_{ih} f_k \right) \frac{\partial g_l}{\partial A_{ij}} \circ D_l \ ,$$

where $s = \sum_j f_j(x_i) \circ D_j$, and $r = \sum_j g_j(x_i A_{ik}) \circ D_j$ on $U \subset E$.

5. 1-jet extension of the natural representation and curvature, Utiyama's theorem

Let us suppose that the manifold V is orientable and endowed with an orientation whose volume element is ω. A *gauge-invariant field* on the fibre bundle of connection $\pi: E \to V$ can be defined as a variational problem (on the 1-jet fibre bundle $J^1(E)$) with a lagrangian density $\mathscr{L}\omega$ admitting the natural representation $\{D_s\}$ of the gauge algebra Γ as a subalgebra of the algebra of infinitesimal internal symmetries [1], i.e., $j^1(D_s)\mathscr{L} = 0$ for every $s \in \Gamma$.

A natural question is now trying to characterize the lagrangians $\mathscr{L}$ satisfying the above said condition. Settled and solved (locally) the problem by Utiyama [8], we want to see, in this section, its geometrical meaning from the point of view of previously introduced notions. In this sense we shall proceed as follows. The curvature 2-form can be interpreted as a mapping $\Omega: J^1(E) \to \bigwedge^2 T^*(V) \otimes L(P)$ by the rule:

$$\Omega(j_x^1\sigma) = (\Omega_\sigma)_x \, .$$

This mapping will be called *curvature mapping*.

Proposition 5.1. *The curvature mapping $\Omega: J^1(E) \to \bigwedge^2 T^*(V) \otimes L(P)$ is an epimorphism of fibre bundles on V, that is, Ω is a differentiable projection making the following diagram commutative:*

$$\begin{array}{ccc} J^1(E) & \xrightarrow{\ \Omega\ } & \Lambda^2 T^*(V) \otimes L(P) \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

Proof. It is obvious that Ω makes the above diagram commutative. Now taking natural local coordinates (x_i, A_{ij}, p_{lmj}) and (x_i, R_{lmj}), $l < m$, on $J^1(E)$ and $\Lambda^2T^*(V) \otimes L(P)$, respectively, the mapping Ω can be written, by using (3.2), as

$$x_i = x_i \, , \qquad R_{lmj} = p_{mlj} - p_{lmj} - \frac{1}{2} \sum_{h,k} c_{hk}^j (A_{lh}A_{mk} - A_{mh}A_{lk}) \, .$$

Thus Ω is differentiable. Now let a point (x_i^0, R_{lmj}^0) be given in $\Lambda^2T^*(V) \otimes L(P)$, let us consider the local section $\sigma: V \to E$ defined by the equations $A_{lj} = \sum_m a_{ml}^j (x_m - x_m^0)$, where a_{ml}^j are arbitrary constants if $m \leq l$ and $a_{ml}^j = a_{lm}^j + R_{lmj}^0$ if $m > l$. σ defines a 1-jet $j_x^1\sigma$ at x such that $\Omega(j_x^1\sigma) = (x_i^0, R_{lmj}^0)$. This proves that Ω is an epimorphism. q.e.d.

On the other hand, s being a given element of the gauge algebra Γ, let X_s be the vertical vector field of the vector bundle $\Lambda^2T^*(V) \otimes L(P)$ such that, for every point $(\omega_2)_x$ and every function f (linear on the fibres) of $\Lambda^2T^*(V) \otimes L(P)$, one has

$$(X_s f)(\omega_2)_x = -f([s(x), (\omega_2)_x]) ,$$

where $[s(x), (\omega_2)_x]$ is the point in $\Lambda^2T^*(V) \otimes L(P)$ defined by

$$[s(x), (\omega_2)_x](D, D') = [s(x), (\omega_2)_x(D, D')] .$$

Proposition 5.2. *The mapping $s \in \Gamma \to D_s$ is a homomorphism of Lie A-algebras.*

Proof. Let $g \in A$ and let f be a function of $\Lambda^2T^*(V) \otimes L(P)$ linear on the fibres. Then

$$\begin{aligned}(X_{gs}f)(\omega_2)_x &= -f([(gs)(x), (\omega_2)_x]) = -f([g(x)\cdot s(x), (\omega_2)_x]) \\ &= -g(x)\cdot f([s(x), (\omega_2)_x]) = g((\omega_2)_x)\cdot (X_s f)(\omega_2)_x = ((gX_s)f)(\omega_2)_x .\end{aligned}$$

This proves that $s \in \Gamma \mapsto X_s$ is A-linear. Now the equality $X_{[s,s']} = [X_s, X_{s'}]$ can be proved in a way analogous to the proof of Theorem 2.1. q.e.d.

In the local coordinate system (x_i, R_{lmj}) in $\Lambda^2T^*(V) \otimes L(P)$ considered before, the vector field X_s is given by

$$X_s = -\sum_{l,m,j,h,k} c^j_{hk} f_h R_{lmk} \frac{\partial}{\partial R_{lmj}} , \tag{5.1}$$

where $s = \sum_j f_j D_j$.

Thus we have two new representations of the gauge algebra Γ: *the* 1-*jet extension $s \in \Gamma \to j^1(D_s)$ of the natural representation and the representation $s \in \Gamma \mapsto X_s$ which we have just defined.* The first is a representation of Γ as a real Lie algebra, and on the other hand the second is a representation of Γ as a Lie A-algebra. This is the essential difference between both representations.

Now Utiyama's theorem can be stated as follows.

Theorem 5.1 (*Utiyama*). *A function $\mathscr{L} : J^1(E) \to R$ is gauge-invariant* (*i.e., it is invariant by the real Lie algebra $\{j^1(D_s) \mid s \in \Gamma\}$*) *if and only if*

$$\mathscr{L} = \bar{\mathscr{L}} \circ \Omega ,$$

where $\bar{\mathscr{L}} : \Lambda^2T^*(V) \otimes L(P) \to R$ is a function invariant by the Lie A-algebra $\{X_s \mid s \in \Gamma\}$, and Ω is the curvature mapping.

Proof. If $s \in \Gamma$ has the local expression $s = \sum_j f_j(x_i)D_j$, then from (2.1) it follows that

$$j^1(D_s) = \sum_j f_j D_j + \sum_{i,j} \frac{\partial f_j}{\partial x_i} D_{ij} + \sum_{(l<m),\, j} \frac{\partial^2 f_j}{\partial x_l \partial x_m} \cdot D_{lmj} ,$$

where

$$D_j = \sum_{i,h,k} c^k_{hj} A_{ih} \frac{\partial}{\partial A_{ik}} + \sum_{l,m,h,k} c^k_{hj} p_{lmh} \frac{\partial}{\partial p_{lmk}} ,$$

$$D_{ij} = \frac{\partial}{\partial A_{ij}} + \sum_{h,k,l} c^k_{hj} A_{lh} \frac{\partial}{\partial p_{ilk}} , \qquad D_{lmj} = \frac{\partial}{\partial p_{lmj}} + \frac{\partial}{\partial p_{mlj}} .$$

Thus $\mathscr{L}$ is gauge-invariant if and only if $\mathscr{L}$ is a solution of the system of (local) equations

$$D_j\mathscr{L} = D_{ij}\mathscr{L} = D_{lmj}\mathscr{L} = 0 .$$

A simple calculus proves that the most general solution of this system is a function $\mathscr{L} = \bar{\mathscr{L}}(x_i, R_{lmj})$ where $\bar{\mathscr{L}}$ satisfies the conditions

$$\sum_{l,m,j,k} c^j_{hk} R_{lmk} \frac{\partial \bar{\mathscr{L}}}{\partial R_{lmj}} = 0 . \tag{5.2}$$

The local coordinates (x_i, A_{ij}, p_{ijk}) and (x_i, R_{lmj}) are defined on $J^1(E)_U$ and $(\Lambda^2 T^*(V) \otimes L(P))_U$ respectively, where U is an arbitrary open set of V with local coordinates (x_i) on which the fibre bundles under consideration trivialize. This, together with the fact that Ω is a fibre bundle epimorphism, implies that the above (local) conditions on $\mathscr{L}$ are equivalent to the (global) condition

$$\mathscr{L} = \bar{\mathscr{L}} \circ \Omega ,$$

where $\bar{\mathscr{L}} : \Lambda^2 T^*(V) \otimes L(P) \to R$ is a function satisfying the (local) conditions (5.2). But, by (5.1), this last fact is equivalent, in turn, to the fact that $\bar{\mathscr{L}}$ be invariant by the Lie A-algebra $\{X_s \mid s \in \Gamma\}$. Thus the theorem is proved. q.e.d.

According to this, gauge-invariant fields on the fibre bundle of connections $\pi : E \to V$ can be parametrized by functions $\bar{\mathscr{L}} : \Lambda^2 T^*(V) \otimes L(P) \to R$ invariant by the Lie A-algebra $\{X_s \mid s \in \Gamma\}$. In particular, it is easy to prove that the following functions are of this type: let p be an arbitrary polynomial of the Weil algebra $H(G)$ of G, and let $F : \Lambda T^*(V) \to R$ be an arbitrary function. We define

$$\bar{\mathscr{L}} : (\omega_2)_x \mapsto F(p(\tilde{\omega}_2)_x) ,$$

where $\sim$ is the canonical injection of $L(P)$-valued forms on V into the $\mathscr{G}$-valued forms on P.

If V is endowed with a pseudo-riemannian metric g we can define a function $\bar{\mathscr{L}}$ of the above type as follows. We take as p the element of $H(G)$ defined by the Cartan-Killing metric on $\mathscr{G}$. Then $p(\tilde{\omega}_2)_x$ is a 4-form on $T_x(V)$, from which we can obtain its scalar square with respect to the metric g, that is,

$$\bar{\mathscr{L}} : (\omega_2) \mapsto g(p(\tilde{\omega}_2)_x, p(\tilde{\omega}_2)_x) .$$

This Lagrange function has been the almost exclusively used one, up to now, in the physics of free gauge-invariant fields.

6. Gauge algebras and external symmetries

To every classical field defined on a fibre bundle $\pi: E \rightarrow V$ by a lagrangian density $\mathscr{L}\omega$ one can associate the extension of real Lie algebras:

$$(6.1) \qquad 0 \longrightarrow \mathscr{D}^v \longrightarrow \mathscr{D} \xrightarrow{\pi} \pi(\mathscr{D}) \longrightarrow 0$$

where $\mathscr{D}$ are the π-projectable vector fields on E such that $L_{j^1(D)}\mathscr{L}\omega = 0$, $\mathscr{D}^v$ is the ideal of vertical vector fields in $\mathscr{D}$, and $\pi(\mathscr{D})$ is the image of $\mathscr{D}$ by the projection π. $\mathscr{D}^v$ and $\pi(\mathscr{D})$ are respectively called "infinitesimal internal symmetries" and "infinitesimal external symmetries" of the field under consideration [1].

Now an important question in classical field theory arises: *how to determine all possible lagrangians such that their corresponding extension* (6.1) (*or part thereof*) *is given in advance*. The problem of Utiyama which we have dealt with in the preceding paragraph, is a typical example of this situation. Nevertheless, in more general situations, it is not likely that such a simple solution can be obtained. In spite of this, it seems that the following general question is a good starting point: Suppose, as it often occurs, that $\mathscr{D}^v$ is the natural representation $\{D_s\}$ of the gauge algebra Γ. *What is the maximal Lie algebra $\mathscr{D}$ having $\mathscr{D}^v$ as an ideal, and what is the corresponding Lie algebra of infinitesimal external symmetries*? By definition, $\mathscr{D}$ is the *idealizator* of $\{D_s\}$ in the Lie algebra of vector fields on the fibre bundle of connections E. The following result gives a very simple answer to this question.

Theorem 6.1. *The idealizator $\mathscr{D}$ of the natural representation $\{D_s\}$ of the gauge algebra Γ in the Lie algebra of vector fields on the fibre bundle of connections $\pi: E \rightarrow V$ coincides with the Lie algebra $\mathscr{H}_\pi$ of hamiltonian π-projectable vector fields on the symplectic manifold (E, Ω). One has the extension of real Lie algebras:*

$$(6.2) \qquad 0 \longrightarrow \{D_s\} \longrightarrow \mathscr{H}_\pi \xrightarrow{\pi} \mathscr{X}(V) \longrightarrow 0$$

where $\mathscr{X}(V)$ are all vector fields on V.

Proof. First of all, $\{D_s\}$ is an ideal of $\mathscr{H}_\pi$, for, if $D_s \in \{D_s\}$, $D_{s'} \in \mathscr{H}_\pi$ and f is a differentiable function on V, then one has

$$[D_s, D_{s'}]f = D_s D_{s'} f - D_{s'} D_s f = 0 ,$$

from which, by the remark to Theorem 4.1 and by Theorem 3.1, one gets $[D_s, D_{s'}] = D_{\{s,s'\}} \in \{D_s\}$. So $\mathscr{H}_\pi \subseteq \mathscr{D}$. On the other hand, it is obvious that $\{D_s\} = \ker \pi|_{\mathscr{H}_\pi}$. Now we go to prove that $\pi(\mathscr{H}_\pi) = \mathscr{X}(V)$, which implies $\mathscr{H}_\pi = \mathscr{D}$ and our result follows.

Let $\sigma_0: V \rightarrow E$ be a connection on the principal bundle P. A vector field D on V being given, let us consider the section $s_D \in \Gamma(L(\bar{P}))$ defined by

$$s_D(\sigma_x) = (\sigma_0)_x D_x - \sigma_x D_x \, , \qquad \sigma_x \in E \, .$$

We want to prove that s_D has a hamiltonian π-projectable vector field $\bar{D}$ such that $\pi(\bar{D}) = D$.

Indeed, let (x_i, A_{ij}) be the local coordinate system on $E_U \subset E$ defined in § 2, and let us suppose that $\sigma_0 : V \to E$ and D have the local expression $A_{ij} = f_{ij}(x_1 \cdots x_n)$ and $D = \sum_i g_i(x_1 \cdots x_n)\partial/\partial x_i$ with respect to (x_i, A_{ij}). Then s_D has the corresponding local expression $s_D = \sum_j \phi_j(x_i, A_{ij}) \circ D_j$, where

$$\phi_j(x_i, A_{ij}) = \sum_j (f_{ij} - A_{ij})g_i \, .$$

Now a simple calculation proves that the equation $i\bar{D}\Omega = ds_D$ has as a (unique) solution the vector field $\bar{D}$ on E whose local expression is

$$\bar{D} = D + \sum_{i,j} \Big[\frac{\partial \phi_j}{\partial x_i} + \sum_{h,k} c^j_{hk} A_{ih} \phi_k - \frac{1}{2} \sum_{l,h,k} c^j_{hk}(A_{lh}A_{ik} - A_{ih}A_{lk})g_l \Big] \cdot \frac{\partial}{\partial A_{ij}} \, . \qquad \text{q.e.d.}$$

In order to illustrate the way in which the above result can be employed, let us consider the following.

Example. Let $p : P \to V$ be the trivial principal bundle $P = R^2 \times U(1)$, and let $\omega = dx_1 \wedge dx_2$ be the euclidean area element of R^2. By Utiyama's theorem, a classical field $\mathscr{L}\omega$ on the fibre bundle of connections $\pi : E \to R^2$ of P admits $\{D_s\}$ as internal symmetries if and only if $\mathscr{L} = \bar{\mathscr{L}} \circ \Omega$, where $\Omega : J^1(E) \to \Lambda^2 T^*(R^2)$ is the curvature mapping and $\bar{\mathscr{L}}$ is an arbitrary function on $\Lambda^2 T^*(R^2)$. Now the question is: what is the relation between $\bar{\mathscr{L}}$ and the external symmetries $\pi(\mathscr{D})$?

By Theorem 6.1, supposing that the extension corresponding to $\mathscr{L}\omega$ is of type

$$(6.3) \qquad 0 \longrightarrow \{D_s\} \longrightarrow \mathscr{D} \xrightarrow{\pi} \pi(\mathscr{D}) \longrightarrow 0$$

we could start our discussion by considering the case of maximum symmetry: $\mathscr{D} = \mathscr{H}_\pi$, $\pi(\mathscr{D}) = \mathscr{X}(V)$. So we must find all functions $\mathscr{L} = \bar{\mathscr{L}} \circ \Omega$ such that, for every $\bar{D} \in \mathscr{H}_\pi$, $L_{j^1(\bar{D})}\mathscr{L}\omega = 0$, that is,

$$(6.4) \qquad j^1(\bar{D})\mathscr{L} + \mathscr{L} \cdot \operatorname{div} \pi(\bar{D}) = 0 \, .$$

By identifying $\Lambda^2 T^*(R^2)$ with $R^2 \times R$ by means of the area element ω, (x_1, x_2, f_{12}) becomes a (global) coordinate system on $\Lambda^2 T^*(R^2)$, f_{12} being the natural coordinate on R, and so we can write $\bar{\mathscr{L}} = \bar{\mathscr{L}}(x_1, x_2, f_{12})$. Now by imposing the invariance condition under $\pi^{-1}(\mathscr{T})$, where $\mathscr{T}$ is the (abelian) Lie algebra of translations of R^2, one has $\bar{\mathscr{L}} = \bar{\mathscr{L}}(f_{12})$ and (6.4) becomes

$$\left(\frac{d\bar{\mathscr{L}}}{df_{12}} - \bar{\mathscr{L}}\right)\cdot \operatorname{div} \pi(\bar{D}) = 0 \ .$$

Taking $\bar{D} \in \mathscr{H}_\pi$ such that $\operatorname{div} \pi(\bar{D}) \neq 0$, one gets $\bar{\mathscr{L}} = \text{const.}\, f_{12}$. This gives us a trivial lagrangian which does define no variational problem. Thus the maximum Lie algebra of infinitesimal symmetries must be, in this example, the set of vector fields D on R^2 such that $\operatorname{div} D = 0$, and the corresponding lagrangian is $\mathscr{L} = \bar{\mathscr{L}} \circ \Omega$ with $\bar{\mathscr{L}} = \bar{\mathscr{L}}(f_{12})$ an arbitrary function. Now we observe that an essential point in the above argument is that the Lie algebra of translations $\mathscr{T}$ is a subalgebra of $\pi(\mathscr{D})$. With this in mind, the rest of discussion can be carried over without difficulties.

Let us now go back to the general case. Another important question in classical field theory is *to determine the splittings of the exact sequence* (6.1). In particular, this allows us to fix the "external Noether invariants" of the field (energy, linear, and angular moments, etc.). In the proof of Theorem 6.1 we see how a connection $\sigma_0 \colon V \to E$ on the principal bundle P determines a splitting (of real vector spaces) $D \in \pi(\mathscr{D}) \to \bar{D}$ of (6.1), $\bar{D}$ being the hamiltonian vector field corresponding to the section s_D defined by the formula

$$s_D(\sigma_x) = (\sigma_0)_x D_x - \sigma_x D_x \ , \qquad \sigma_x \in E \ . \tag{6.5}$$

In general, this splitting does not preserve Lie brackets. Now an interesting question is to characterize those connections whose corresponding splittings preserve Lie brackets. This would give us in particular, a differential-geometric procedure of "mixing" gauge algebras and external symmetries very close to the physical problem. The following result gives an answer to this question.

Theorem 6.2. *Let $\sigma_0 \colon V \to E$ a connection on the principal bundle P with 2-form of curvature Ω, let D, D' be two vector fields on V, and let $s_D, s_{D'} \in \bar{\Gamma}$ be the sections defined by D, D' according to the above formula* (6.5). *Then*

$$s_{[D,D']} = \{s_D, s_{D'}\} - \Omega(D, D') \ . \tag{6.6}$$

Thus σ_0 defines a splitting (of real Lie algebras) of the exact sequence (6.1) *if and only if $\Omega(D, D') \in \Gamma_0$ for every pair D, D' of infinitesimal external symmetries. In particular, this is true if σ_0 is a flat connection.*

Proof. It will be enough to compute the Poisson bracket $\{s_D, s_{D'}\}$ having in mind the local expression for the 2-form of curvature of a connection. For the last part, it is enough to remember Theorem 4.1(c). q.e.d.

According to this, existence of splittings of (6.1) induced by connections should, in general, influence the principal bundle P and, eventually, the splitting itself. Thus, for example, if the splitting is induced by a flat connection, σ_0, and the base manifold V is simply connected, then P must be isomorphic with the trivial bundle $V \times G$ and σ_0 is isomorphic with the canonical flat connection on $V \times G$ [5]. Then the exact sequence (6.1) has, up to equivalences, a unique splitting, which coincides, in the particular cases dealt with in physics, with the "trivial combination" of gauge algebras and external symmetries. Simmilarity of this result and O'Raifertaigh's theorems [7] forbidding nontrivial combinations of "internal (finite-dimensional)" and "space-time" symmetries is well apparent. This remark could be a starting point for a differential-geometric approach to this interesting topic for infinite dimensional Lie algebras of the type which this paper deals with.

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